An Analysis of a Time-Dependent Multi-Server Queueing Model with Feedback and Reneging

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ABSTRACT

In this paper, an infinite waiting space markovian multi-server feedback queueing problem with reneging is studied. Arrival process follows the Poisson distribution, inter-arrival time and service time have exponential distribution. In addition to this, some customers may renege with a fixed probability. Using Laplace transformation technique, probability generating function of transient state queue length probabilities is obtained and some special cases have been derived as well. Finally, the graphical solution of the problem is discussed.

Keywords: Multi-server, feedback, impatient customers, Laplace transform, transient-state, generating function, exponential distribution.

INTRODUCTION

The queueing theory is mostly concerned with queueing systems in which all customers leave the system after getting their service. There is little work available on queueing system with impatient customers. Haight [5] was the first to study the queueing problem with reneging in which he studied the problem like how to make a rational decision while waiting in the queue, the probable effect of this decision etc. Ancker and Gafarian[2], Zhang et.al.[11], Jain and Singh[6], Jindalet.al [7] and Choudhury& Medhi[3] studied a number of queueing problems with impatient customers.

Markovian queueing problems with feedback have also achieved considerable attention by several researchers. In Feedback, the customer gets additional service if they are not satisfied with their previous ill-mannered service. In this respect, we can refer to Tackas’[9] Pioneering work. Scharge [8] studied “The queue M/G/1 with feedback to lower priority queues”. Laplace transforms and expression for the moments of the time in system distribution are obtained. Thangaraj and Vanitha [10] also studied feedback queueing problems and time dependent probability generating function in terms of Laplace transformation; mean queue length and waiting time are obtained.

Multi-server queueing system has applications in many real life situations such as in hospitals, banks, retail shops; hotel-management etc. for the longer queue, the waiting time of the customers is reduced by using multi-server. Garg and Kumari[4] studied multi-server feedback queueing problem and transient-state queue length probabilities are obtained. Alseedy et. al. [1] also studied an M/M/c queue with balking & reneging and obtained the transient solution by using probability generating function & Bessel function.

In this paper, the analysis of M/M/C feedback queueing problem with reneging under transient-state is presented. Wherein the customers arrive and join any of the free servers and after being served once will leave the system if satisfied from the service and re-join the queue if they are not satisfied from their previous service. The customers however leave the system definitely after having received the service for second time. When the numbers of customers are more than the number of server then the arriving customers join the waiting line and wait for their service. Sometimes a customer leaves the queue without getting service due to impatience. Laplace transform of the probability generating function of transient-state queue length probabilities and the graphical solution of the problem are discussed.

The queueing system investigated in this paper is governed by the following assumptions

1) Arrivals are Poisson with parameter λ and service time distribution of every customer for each of the C servers is exponential with parameter μ.

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2) The probability of re-joining the customer to the system is ‘p’ and that of leaving the system is ‘q’ for the customers getting first service, so that \( p + q = 1 \). However, the customers will have to leave the system after getting second service.

3) After the current departure, the next customer will depart the service channel for the first time with probability \( c_1 \) and for the second time with probability \( c_2 \), so that \( c_1 + c_2 = 1 \).

4) The probability of a customer reneging during time \( \Delta t \), when there are \( n \) customers in the queue is \( r(n) \Delta t \) and also assumed that reneging follow the exponential distribution with density function

\[
d(t) = \alpha e^{-\alpha t}
\]

Also \( r(n)=\begin{cases} 0, & 0 < n \leq c \\ \alpha, & n > c \end{cases} \)

5) The waiting space is infinite.

6) The stochastic processes involved, viz

a) Arrival of units
b) Departure of units

are statistically independent.

**Definitions**

\( P_n^{(0)}(t) \) = Probability that there are \( n \) customers in the system at time \( t \) and next customer is to depart for the first time.

\( P_n^{(1)}(t) \) = Probability that there are \( n \) customers in the system at time \( t \) and next customer is to depart for the second time.

\( P_n(t) \) = Probability that there are \( n \) customers in the system at time \( t \)

\[
P_n(t) = P_n^{(0)}(t) + P_n^{(1)}(t) \quad n \geq 0
\]

Initially

\[
P_0^{(0)}(0) = 1 \quad \text{and} \quad P_0^{(1)}(t) = 0, t \geq 0
\]

**The difference – differential equations describing the system are**

\[
\frac{d}{dt} P_n^{(0)}(t) = -\lambda P_n^{(0)}(t) + \alpha P_n^{(1)}(t) + \alpha P_{n+1}^{(1)}(t)
\]

\[
\frac{d}{dt} P_n^{(1)}(t) = -\lambda P_n^{(1)}(t) + \lambda P_{n-1}^{(1)}(t) + \alpha P_n^{(1)}(t) + (n + 1) \alpha c_1 P_{n+1}^{(1)}(t)
\]

\[
\frac{d}{dt} P_n^{(0)}(t) = -\lambda P_n^{(0)}(t) + \lambda P_{n-1}^{(0)}(t) + \alpha P_n^{(0)}(t) + (n + 1) \alpha c_2 P_{n+1}^{(0)}(t) + (n + 1) \alpha p c_1 P_{n+1}^{(1)}(t)
\]

\[
\frac{d}{dt} P_n^{(1)}(t) = -\lambda P_n^{(1)}(t) + \lambda P_{n-1}^{(1)}(t) + \alpha P_n^{(1)}(t) + (n + 1) \alpha c_2 P_{n+1}^{(0)}(t) + \alpha p c_1 P_{n+1}^{(1)}(t)
\]

Where \( \delta_{n,i} = \begin{cases} 1, & \text{for } n = i \\ 0, & \text{otherwise} \end{cases} \)

**Taking the Laplace transformation** \( \tilde{P}_n(s) = \int_0^\infty e^{-st} P_n(t) dt \); \( \text{Re } s > 0 \) of (2) – (6)

\[
(s + \lambda) \tilde{P}_n^{(0)}(s) = 1 + \alpha \tilde{P}_n^{(1)}(s) + \alpha \tilde{P}_n^{(1)}(s)
\]
(s + \lambda + n\overline{\mu} \overline{P}^{(0)}_n(s)
= \lambda \overline{P}^{(0)}_{n-1}(s) + n\overline{\mu} p_1 (1 - s) \overline{P}^{(0)}_n(s) + (n + 1) \mu q c_1 \overline{P}^{(0)}_{n+1}(s)
+ (n + 1) \mu q c_1 \overline{P}^{(1)}_{n+1}(s)
1 \leq n \leq c - 1 \ (8)

(s + \lambda + n\overline{\mu} \overline{P}^{(1)}_n(s)
= \lambda \overline{P}^{(1)}_{n-1}(s) + (n + 1) \mu q c_2 \overline{P}^{(1)}_{n+1}(s) + n\overline{\mu} p_1 (1 - s) \overline{P}^{(0)}_n(s)
+ (n + 1) \mu q c_2 \overline{P}^{(1)}_{n+1}(s)
1 \leq n \leq c - 1 \ (9)

\{s + \lambda + c\overline{\mu} + n\overline{\mu} \overline{P}^{(0)}_n(s)
= \lambda \overline{P}^{(0)}_{n-1}(s) + n\overline{\mu} p_1 (1 - s) \overline{P}^{(0)}_n(s) + (n + 1) \mu q c_1 \overline{P}^{(0)}_{n+1}(s)
+ (n + 1) \mu q c_2 \overline{P}^{(1)}_{n+1}(s)
1 \leq n \leq c \ (10)

\{s + \lambda + c\overline{\mu} + n\overline{\mu} \overline{P}^{(1)}_n(s)
= \lambda \overline{P}^{(1)}_{n-1}(s) + (n + 1) \mu q c_1 \overline{P}^{(1)}_{n+1}(s) + n\overline{\mu} p_1 (1 - s) \overline{P}^{(0)}_n(s)
+ (n + 1) \mu q c_2 \overline{P}^{(1)}_{n+1}(s)
1 \leq n \leq c \ (11)

Define

\[ P^{(0)}(z, t) = \sum_{n=0}^{\infty} P^{(0)}_n(t) z^n \overline{P}^{(0)}(z, s) = \int_0^{\infty} e^{-st} P^{(0)}(z, t) dt \]

\[ P^{(1)}(z, t) = \sum_{n=0}^{\infty} P^{(1)}_n(t) z^n \overline{P}^{(1)}(z, s) = \int_0^{\infty} e^{-st} P^{(1)}(z, t) dt \]

\[ P(z, t) = P^{(0)}(z, t) + P^{(1)}(z, t) \]

With \(|z| \leq 1\)

Laplace transformation of probability generating function of transient – state queue length probabilities

\[ \mu z (A - c\overline{\mu}) \left\{ c_2 \overline{P}^{(1)}_l (s) + (c_2 q - c_1 p z) \overline{P}^{(0)}_l (s) \right\} \]

\[ + \sum_{n=0}^{c} \left\{ (A - c\mu c_2) \{ -\mu z (n - c) - \alpha (1 - z) \} + (n - c) A \mu c_1 (q + p z) \} \overline{P}^{(0)}_n (s) z^n \]

\[ + \sum_{n=0}^{c} \mu c_1 \{ (A - c\mu z) (n - c) - c\alpha (1 - z) \} \overline{P}^{(1)}_n (s) z^n + z (A - c\mu c_2) \]

\[ \overline{P}^{(0)}(z, s) = \frac{(A - c\mu c_2) \{ A - c\mu c_1 (q + p z) \} - c^2 \mu^2 c_1 c_2 (q + p z)}{\lambda < c\mu, |z| \leq 1} \ (12) \]
\[-\mu z(A - c\mu(q + pz))\left\{c_2 P_1^{(1)}(s) + (c_2q - c_1pz)\bar{P}_1^{(0)}(s)\right\} \\
+ \sum_{n=0}^{\infty} \mu c_2(q + pz)\{(A - c\mu z)(n - c) - c\alpha(1 - z)\}\bar{P}_n^{(0)}(s)z^n \\
+ \sum_{n=0}^{\infty} \{(A - c\mu c_1(q + pz))\{-\mu (n - c) - \alpha(1 - z)\} + (n - c)A\mu c_2\} \bar{P}_n^{(1)}(s)z^n \\
+ zc\mu c_2(q + pz)\]

\[
\bar{P}^{(1)}(z, s) = \frac{A - c\mu c_2(A - c\mu c_1(q + pz)) - c^2\mu^2 c_1 c_2(q + pz)}{\lambda < c\mu, |z| \leq 1} (13)
\]

\[
(1 - z)\left[-c\mu^2pz\left\{c_2 P_1^{(1)}(s) + (c_2q - c_1pz)\bar{P}_1^{(0)}(s)\right\} \\
+ \sum_{n=0}^{\infty} \{(n - c)(A\mu q + c\mu^2 c_2pz) - \alpha(A - c\mu c_2p(1 - z))\}\bar{P}_n^{(0)}(s)z^n \\
+ \sum_{n=0}^{\infty} \{(n - c)(A\mu - c\mu^2 c_1pz) - \alpha(A + c\mu c_1 p(1 - z))\} \bar{P}_n^{(1)}(s)z^n \right] + z(A - c\mu c_2p(1 - z))
\]

\[
\bar{P}(z, s) = \frac{(A - c\mu c_2)(A - c\mu c_1(q + pz)) - c^2\mu^2 c_1 c_2(q + pz)}{\lambda < c\mu, |z| \leq 1} (14)
\]

Where \(A = -\lambda z^2 + (s + \lambda + c\bar{\mu} + \alpha)z - \alpha\)

Let \(D = K_1(z)K_2(z) - c^2\bar{\mu}^2 c_1 c_2(q + pz)\)

Where \(K_1(z) = (-\lambda z^2 + (s + \lambda + c\bar{\mu} + \alpha)z - (c\bar{\mu} c_2 + \alpha)\)

\(K_2(z) = (-\lambda z^2 + (s + \lambda + c\bar{\mu} + \alpha - c\bar{\mu} c_1 p)z - (c\bar{\mu} c_1 q + \alpha)\)

Obviously \(K_1(z)\) and \(K_2(z)\) have two zeros inside the unit circle.

Let \(f(z) = K_1(z)K_2(z)\) and \(g(z) = c^2\bar{\mu}^2 c_1 c_2(q + pz)\)

\[
|f(z)| = |K_1(z)K_2(z)| = \left|(-\lambda z^2 + (s + \lambda + c\bar{\mu} + \alpha)z - (c\bar{\mu} c_2 + \alpha)\right| \\
\left|(-\lambda z^2 + (s + \lambda + c\bar{\mu} + \alpha - c\bar{\mu} c_1 p)z - (c\bar{\mu} c_1 q + \alpha)\right| \\
\geq (\xi + \bar{c}\bar{\mu} c_1)(\xi + c\bar{\mu} c_2)\]

For \(s = \xi + i\eta, |z| = 1\)

\[
\geq c^2\bar{\mu}^2 c_1 c_2 \geq |g(z)|
\]

Hence \(|f(z)| \geq |g(z)|\) on \(|z| = 1\)

Since all the condition of Rouche's theorem are satisfied, so \(D\) has two zeroes inside the unit circle. Let these zeroes be \(z_m (m = 0, 1)\). Numerator must also vanish for these two zeroes since \(\bar{P}(z, s)\) is an analytical function of \(z\). These two equations along with equation (7) and equations \((8)\&(9)\) for \(n = 1, 2, 3, \ldots \ldots, c - 1\) will in general determine \((2c + 1)\) unknowns \(\bar{P}_0^{(0)}(s), \bar{P}_1^{(0)}(s), \bar{P}_1^{(1)}(s), \ldots \ldots, \bar{P}_c^{(0)}(s), \bar{P}_c^{(1)}(s), \bar{P}_{c-1}^{(0)}(s), \bar{P}_{c-1}^{(1)}(s), \bar{P}_c^{(1)}(s), \bar{P}_{c-1}^{(1)}(s)\). Hence the generating function \(\bar{P}(z, s)\) is completely known.

\(\bar{P}_n(s)\) Can be obtained by using the following formula

\[
\bar{P}_n(s) = \frac{1}{n!} \frac{d^n(\bar{P}(z, s))}{dz^n} \quad \text{at} \quad z = 0
\]

And \(\bar{P}_n(t)\) can be found by inverting the Laplace transform \(\bar{P}_n(s)\).

Further
\( \bar{P}(1, s) = \frac{1}{s} \), as desired

And

\( \bar{P}(0, s) = \bar{P}^{(0)}(s) \)

Special cases

1) **When there is no feedback.**

Putting \( q = 1, \ p = 0, c_1 = 1, c_2 = 0, P^{(1)}(z, s) = 0, \bar{P}^{(0)}(z, s) = \bar{P}(z, s), \bar{P}_n^{(0)}(s) = \bar{P}_n(s), \bar{P}_n^{(1)}(s) = 0 \) in equation (14) we get

\[
\bar{P}(z, s) = \frac{z + (1 - z) \sum_{n=0}^{c} (n - c)(\mu - \alpha) \bar{P}_n(s) z^n}{\lambda z^2 + (s + \lambda + c\mu + \alpha)z - (c\mu + \alpha)}
\]

(15)

2) **When there is no feedback and no reneging.**

Put \( \alpha = 0 \) in equation (15) we get

\[
\bar{P}(z, s) = \frac{z + (1 - z) \sum_{n=0}^{c} (n - c)\mu \bar{P}_n(s) z^n}{-\lambda z^2 + (s + \lambda + c\mu)z - c\mu}
\]

(16)

3) **When there is no feedback and no reneging and number of servers is one.**

Put \( c = 1 \) in equation (16) we get

\[
\bar{P}(z, s) = \frac{z - (1 - z)\mu \bar{P}_0(s)}{-\lambda z^2 + (s + \lambda + \mu)z - \mu}
\]

(17)

This coincides with the generating function of transient- state M/M/1 model.

**GRAPHICAL SOLUTIONS**

Various probabilities are plotted vs. time for the data obtained by using Matlab programming. Fig 1.1 shows plot of probability \( P_0^{(0)} \) with respect to time \( t \) (average service time). It is clear from the graph that probability \( P_0^{(0)} \) decreasing rapidly in the starting and then becomes almost steady from the initial value (at time \( t=0 \)).
Figs 1.3 and 1.4 show relative change in probabilities $P_3^0, P_3^1, P_4^0, P_4^1, P_5^0, P_5^1, P_6^0, P_6^1, P_7^0$, and $P_7^1$ with respect to time (average service time). Probability $P_3^0, P_4^0, P_5^0, P_6^0, P_7^0$ increases rapidly in the starting then starts decrease and finally attain some steady values for higher values of $t$. Though the probabilities $P_3^1, P_4^1$ (shown in fig 1.3) and $P_5^1, P_6^1, P_7^1$ (shown in Fig1.4 increase in the starting but the increase comparatively less than their corresponding counterparts $P_3^0, P_4^0, P_5^0, P_6^0$ and $P_7^0$. These also attain some steady values for higher values of $t$. From the above interpretations we can say that the probability of joining the server for the first time is more than that of joining server for the second time.
To study the effect of reneging on different probabilities of the model, the data of various probabilities is generated for different values of $\alpha$ keeping the other parameters constant. The values that $\alpha$ took are $\{0.2 \& 0.5\}$. The other parameters are fixed at $\lambda=2$, $\mu=1$, $c=3$, $c_1=0.8$, $q=0.75$. Fig. 1.5 and Fig. 1.6 concluded that as $\alpha$ increase probabilities $P_4(0)$, $P_5(0)$, $P_6(0)$, $P_7(0)$, $P_4(1)$, $P_5(1)$, $P_6(1)$, $P_7(1)$ are decreasing.
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